Linear 1^{ST} Order Equations – Special Case

Bernoulli Eq'n: format $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Linear if $n=0$ or $n=1$

otherwise use $v = y^{1-n}$

revisit $2xy\frac{dy}{dx} = 4x^2 + 3y^2$ ← homogeneous eq'n

↓ rewrote as

$$\frac{dy}{dx} = \frac{2x}{y} + \frac{3}{2}\frac{y}{x}$$

$$\frac{dy}{dx} \underbrace{- \frac{3}{2}\frac{y}{x}}_{P(x) = -\frac{3}{2x}} = \frac{2x}{y} \quad Q(x)$$

y^{-1} ← $n=-1$

$$v = y^{1-(-1)} = y^2 \Rightarrow y = v^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$$

$$\frac{1}{2}v^{-1/2}\frac{dv}{dx} - \frac{3}{2x}\cdot v^{1/2} = 2xv^{-1/2}$$

$$\frac{1}{2}\frac{dv}{dx} - \frac{3}{2x}v = 2x \quad \text{← now in linear format}$$

$$\frac{dv}{dx} \underbrace{- \frac{3}{x}}_{P(x)} v = 4x$$

$$\rho = e^{-3\int\frac{1}{x}dx} = x^{-3}$$

$$x^{-3}\frac{dv}{dx} - \frac{3}{x}x^{-3}v = 4x(x^{-3})$$

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

$$D_x[v\cdot x^{-3}] = 4x^{-2}$$

$$v\cdot x^{-3} = 4\int x^{-2}\,dx$$

$$v\cdot x^{-3} = -\frac{4}{x} + C \quad \text{where } C \text{ is arb. constant}$$

$$y^2\cdot x^{-3} = -\frac{4}{x} + C$$

$$\boxed{y^2 = -4x^2 + Cx^3} \quad \text{← same as previous}$$

ex. $2xe^{2y}\dfrac{dy}{dx} = 3x^4 + e^{2y}$

- not separable
- not linear
- not Bernoulli

however $\dfrac{d}{dx}\left(e^{2y}\right) = 2e^{2y}\dfrac{dy}{dx}$

then $v = e^{2y}$

$\dfrac{dv}{dx} = 2e^{2y}\dfrac{dy}{dx}$

$x \cdot 2e^{2y}\dfrac{dy}{dx}$

$x \cdot \dfrac{dv}{dx} = 3x^4 + v$

$x\dfrac{dv}{dx} - v = 3x^4$

$\dfrac{dv}{dx} - \dfrac{v}{x} = 3x^3$ $\quad P(x) = -\dfrac{1}{x} \qquad \rho = \dfrac{1}{x}$

$\dfrac{1}{x}\dfrac{dv}{dx} - \dfrac{1}{x^2}v = 3x^2$

$D_x\left[v \cdot \dfrac{1}{x}\right] = 3x^2$

$v \cdot \dfrac{1}{x} = x^3 + C$ where C is arb. constant

$e^{2y}\dfrac{1}{x} = x^3 + C$

$\boxed{e^{2y} = x^4 + Cx}$ or $y(x) = \frac{1}{2}\ln\left|x^4 + Cx\right|$

Reducible Second Order Equations

contains 2nd derivative

format: $F(x, y, y', y'') = 0$

when y is missing:

$F(x, y', y'') = 0$

rewrite

$\boxed{p = y' = \dfrac{dy}{dx}}$ then $\boxed{y'' = \dfrac{dp}{dx}}$

$F(x, p, p') = 0$

ex. solve $xy'' + 2y' = 6x$

$x\dfrac{dp}{dx} + 2p = 6x$

$$x\frac{dp}{dx} + 2p = 6x$$

$$\frac{dp}{dx} + \frac{2}{x}p = 6$$

$P(x) \longrightarrow \rho = x^2$

$$x^2\frac{dp}{dx} + 2xp = 6x^2$$

$$D_x[p\cdot x^2] = 6x^2$$

$$p\cdot x^2 = 2x^3 + C \quad \text{where } C \text{ is arb. constant}$$

$$\frac{dy}{dx}x^2 = 2x^3 + C$$

$$\frac{dy}{dx} = 2x + \frac{C}{x^2}$$

integrate: $y(x) = x^2 + C\int x^{-2}\,dx + D$ where D is arb. constant

$$\boxed{y(x) = x^2 - \frac{C}{x} + D}$$

when x is missing: $F(y, y', y'') = 0$

let $y' = p$, $y' = \frac{dy}{dx}$, $y'' = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$, $\frac{dy}{dx} = p$

rewrite

$$F\left(y, p, p\frac{dp}{dy}\right) = 0$$

ex. solve $yy'' = (y')^2$ given $y > 0$ and $y' > 0$

$$yp\frac{dp}{dy} = p^2 \quad \text{separable :)}$$

$$\frac{1}{p}dp = \frac{1}{y}dy$$

$$\ln p = \ln y + C \quad \text{where } C \text{ is arb. constant}$$

$$p = e^C y \quad \text{let } A = e^C \text{ always } \oplus$$

$$p = Ay$$

$P = e^{C} y$ let $A = e^{C}$ ← always ⊕

$P = A y$

$\frac{dy}{dx} = Ay$ separable

$\frac{1}{y} dy = A\, dx$

$\ln y = Ax + D$ where D is arb. constant

$y = e^{D} e^{Ax}$ let $B = e^{D}$ ← always +

$$\boxed{y(x) = B e^{Ax}}$$

Section 2.1 Population Models

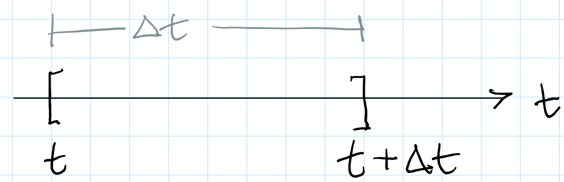
population format recall: $\frac{dP}{dt} = kP$ has sol'n $P(t) = P_0 e^{kt}$

represents natural population growth that occurs from perpetual births and deaths

rates (not necessarily constant)

$\beta(t)$ birth rate

$\delta(t)$ death rate



random interval of time: $[t, t+\Delta t]$

then births during $[t, t+\Delta t]$ would be: $\underset{\text{rate}}{\beta(t)} \cdot \underset{\text{"value"}}{P(t)} \cdot \underset{\text{length}}{\Delta t}$

likewise deaths would be: $\delta(t) \cdot P(t) \cdot \Delta t$

then change in population on $[t, t+\Delta t]$:

$$\Delta P = [\text{births}] - [\text{deaths}]$$

$$\Delta P \approx \beta(t) \cdot \underline{P(t)} \cdot \Delta t - \delta(t) \cdot \underline{P(t)}\, \Delta t$$

$$\frac{\Delta P}{\Delta t} \approx [\beta(t) - \delta(t)]\, P(t)$$

limit $\frac{dP}{dt} = (\beta - \delta)\, P$ $\beta(t) = \beta$, etc. for brevity limit as $\Delta t \to 0$ $\quad \frac{dP}{dt} = \underbrace{(\beta - \delta)}_{\text{become } k \text{ when } \beta, \delta \text{ are constant}} P \qquad \beta(t) = \beta$, etc. for brevity

"expanded" def'n opens up possibility that birth/death rates can be either constant or variable

ex. initial alligator population is 100
none are dying *

if $\beta = (0.0005)P$ then initial value problem would be

$$\frac{dP}{dt} = [(0.0005)P - \delta]P \qquad \delta = 0 \ * \qquad \text{where } P(0) = 100$$

$$\frac{dP}{dt} = 0.0005P^2 \qquad \text{separable}$$

$$\frac{1}{P^2}\,dP = 0.0005\,dt$$

$$\int P^{-2}\,dP = 0.0005\int dt$$

$$-\frac{1}{P} = 0.0005t + C \qquad \text{where } C \text{ is arb. constant}$$

$$0.0005 = \frac{5}{10(1000)} = \frac{1}{2000}$$

when $P(0) = 100$: $-\frac{1}{100} = C$
(t, P)

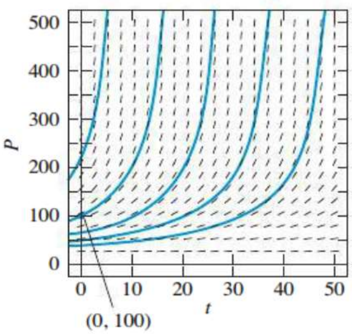
$$P\left(-\frac{1}{P} = \frac{1}{2000}t - \frac{1}{100}\right)$$

$$2000\left[-1 = P\left(\frac{1}{2000}t - \frac{1}{100}\right)\right]$$

$$-2000 = P(t - 20)$$

$$P(t) = -\frac{2000}{t-20}$$

$$\boxed{P(t) = \frac{2000}{20-t}}$$



after 10 years: $P(10) = \frac{2000}{10} = 200$ (doubled)

population is unbounded for a finite amount of time

Bounded Populations / Logistic Equation

population is unbounded for a finite amount of time

Bounded Populations / Logistic Equation

in some cases, although birth rate decreases, pop'n still INCREASES

ex. β is decreasing linear function of P s.t. $\beta = \beta_0 - \beta_1 P$

where β_0, β_1 are ⊕ constants

also δ is constant s.t. $\delta = \delta_0$

then
$$\frac{dP}{dt} = (\beta - \delta)P$$
$$= (\beta_0 - \beta_1 P - \delta_0)P$$
$$= (a - bP)P$$

let $a = \beta_0 - \delta_0$
$b = \beta_1$

$$\boxed{\frac{dP}{dt} = aP - bP^2}$$

assuming a, b are both ⊕ then called logistic equation

rewrite in more familiar form:

let $k = b$
$M = \frac{a}{b}$

$\therefore a = Mb$

$$\frac{dP}{dt} = M\cancel{b}^{k}P - kP^2$$

$$\frac{dP}{dt} = kP(M - P)$$

where M is carrying capacity

(maximum pop'n that environment is capable of sustaining long term)